Coupled synchronization of spatiotemporal chaos

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The synchronization of spatiotemporal chaos is very important in secure communication. In this paper, we present an approach of nonlinear coupling to implement the synchronization of spatiotemporal chaos. Two cases are considered. The first is the same systems with same parameters; the second is the same systems with different parameters. And the largest Lyapunov exponent spectra are calculated. Our numerical simulation shows that the mutual coupling can induce generalized synchronization. [S1063-651X(99)02302-8]

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I. INTRODUCTION

Chaos synchronization has recently aroused a great deal of interest in the light of potential applications in engineeing [1]. Techniques based on the Pecora-Carroll method have been very successful for synchronizing chaos in low-dimensional systems [2–6]. People extended the concept of identical synchronization (IS) to generalized synchronization (GS) [7–9]. It equates dynamical variables from subsystem with a function of the variables of another subsystem and exists in directionally coupled chaotic system. Synchronizing spatiotemporal systems remains a challenge [10–14], however, because the chaotic states in such systems are typically high dimensional, involving multiple stable and unstable modes. There are some great advantages of spatiotemporal chaos in comparison with low-dimensional chaos. The information operations can be performed simultaneously and in parallel by many subunits if one can properly drive and control extended systems, and thus the efficiency of information treatment can be significantly enhanced. The potential for the applications of spatiotemporal chaos control and synchronization is extremely great and unlimited.

Recently, researchers have studied the collective behavior of systems consisting of a large number of coupled identical units [10,15–18]. Such models may represent active networks with elements located in the junctions of a discrete space lattice. Lai and Winslow [19] studied the dynamics of spatiotemporal chaotic systems described by systems of coupled Hénon maps and coupled Duffing equations, and found the extreme sensitivity of asymptotic attractors to both initial conditions and parameters. In this paper we consider a well-known coupled map lattice (CML) model [20,21],

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\tfrac{1}{2}\epsilon[f(x_n(i-1))+f(x_n(i+1))], \tag{1}$$

where $i=1,2,\ldots N$ are the lattice sites, and N the system size. And periodic boundary conditions, $x_n(i+N)=x_n(i)$, is assumed. Moreover, we take $f(x)=ax(1-x)$. With single site ($N=1$), model (1) reduces to the well-known Logistic map, which has a period-doubling cascade with the accumulation point at $a_c=3.569\,945\,6\ldots$, and chaos can be found in the regime $a_c<a<4$. The dynamical behavior of model (1) with $N>1$ has been also extensively investigated and quite well understood. It has many positive Lyapunov exponents when N is large. And the number of positive Lyapunov exponents will increase with the increasing of the system size N. In this paper we will present a nonlinear coupling approach to implement the mutual synchronization of two CML systems. We here prove the possibility of the mutual synchronization of regular or chaotic space patterns of two interacting lattices. We consider two kinds of synchronization in two coupled map lattice dynamical systems. The first is the same CML models with the same parameters, the second is the same CML models with different parameters. We find that the GS can be obtained in these cases.

We have organized this paper as follows. In Sec. II a brief description of the nonlinear coupled algorithm and some results of numerical experiments are presented. Then, in Sec. III, the second case is discussed. Finally, Sec. IV gives the explanation of the mutual synchronization by Lyapunov exponent spectra and some conclusions.

II. COUPLED SYNCHRONIZATION IN THE SAME CML MODELS WITH THE SAME PARAMETERS

Consider two of the same CML models (1) with different initial points, where one is a $x(i)$ system and another a $y(i)$ system. The lattice length is taken to be $N=60$, and the initial condition is prepared as pseudorandom numbers uniformly distributed in the interval [0,1]. Afterwards, we will always take $N=60$ and use such an initial condition unless specified otherwise. Here we take

$$f(x(i))=ax(i)[1-x(i)],$$

$$f(y(i))=ay(i)[1-y(i)], \quad i=1,2,\ldots,N. \tag{2}$$

Substituting Eq. (2) into Eq. (1) one gets a pair of systems with N dimensional. These lattice systems representing spatially extended dynamical systems may exhibit attractors which are chaotic not only in time but also in space. Figure 1 shows the phase portrait of no coupling between the two CML models for different initial points when $a=3.7$, where $x(i)$ represents all of the values of N sites.

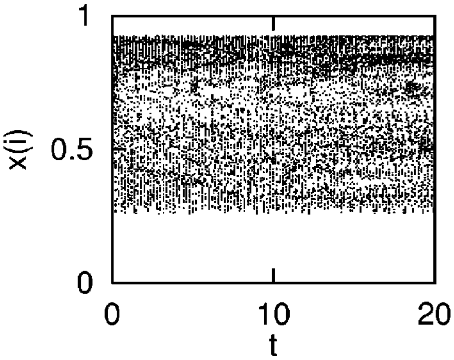

FIG. 1. The phase portrait of no coupling, where $a=3.7$ and the initial condition is prepared as pseudo-random numbers uniformly distributed in the interval [0,1].

To synchronize these systems, we use nonlinear coupling in the following way:

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i)) + \tfrac{1}{2}\epsilon[f(x_n(i-1))+f(x_n(i+1))]+g_{xn}(i),$$

$$y_{n+1}(i)=(1-\epsilon)f(y_n(i)) + \tfrac{1}{2}\epsilon[f(y_n(i-1))+f(y_n(i+1))]+g_{yn}(i) \quad (3)$$

with

$$g_{xn}(i)=(1-\epsilon)px_n(i)[x_n(i)-y_n(i)) + \tfrac{1}{2}\epsilon p\{x_n(i-1)[x_n(i-1)-y_n(i-1)] + x_n(i+1)[x_n(i+1)-y_n(i+1)]\},$$

$$g_{yn}(i)=(1-\epsilon)py_n(i)[y_n(i)-x_n(i)] + \tfrac{1}{2}\epsilon p\{y_n(i-1)[y_n(i-1)-x_n(i-1)] + y_n(i+1)[y_n(i+1)-x_n(i+1)]\}, \quad (4)$$

where p is the coupling strength, ϵ a constant in [0,1]. Throughout this paper we take $\epsilon=0.8$. Now we consider the evolution of difference $y_n(i)-x_n(i), i=1,2,\ldots N$. Making linear approximation, from Eqs. (3) and (4) we have

$$y_{n+1}(i)-x_{n+1}(i)=Z_{ij}\delta y_n(j), \quad i,j=1,2,\ldots N. \quad (5)$$

When we choose a suitable parameter p, we can let all of the absolute values of eigenvalues of $|Z|$ become smaller than unit, and then the two lattice systems can become synchronized. Otherwise, the synchronization will not appear. Figure 2 shows the results when $p=0.87$, where (a) and (b) represent the evolution of $x(i)$ and $y(i)$ systems, respectively, and (c) error dynamics between $x(i)$ and $y(i)$ systems. Comparing Fig. 2(a) with Fig. 2(b) one can find that they have similar behavior when $t>20$. However, Fig. 2(c) shows that the differences $y(i)-x(i)$ do not go to zero. What is the reason for this phenomenon? Testing one couple of sites of the systems, we find that they become time-period-four when the synchronization is implemented. Figure 3 shows the local structure. It is interesting to find that from Fig. 3(c) one can see that the difference $y(10)-x(10)$ is near zero when $t<10$ and suddenly becomes time-period-four when $t>15$. It illustrates that the two corresponding sites is becoming IS when $t<10$. That is because, the coupling between these maps is nonlinear, but at the same time it is proportional to differences, $y(i)-x(i)$. Therefore there exists an identity synchronization manifold on which $x_n(t)=y_n(t)$ is satisfied for some values of p. Thus we expect the two chains to be identicaly synchronized and still remain spatiotemporally chaotic. However, with the proceeding of coupling, the two

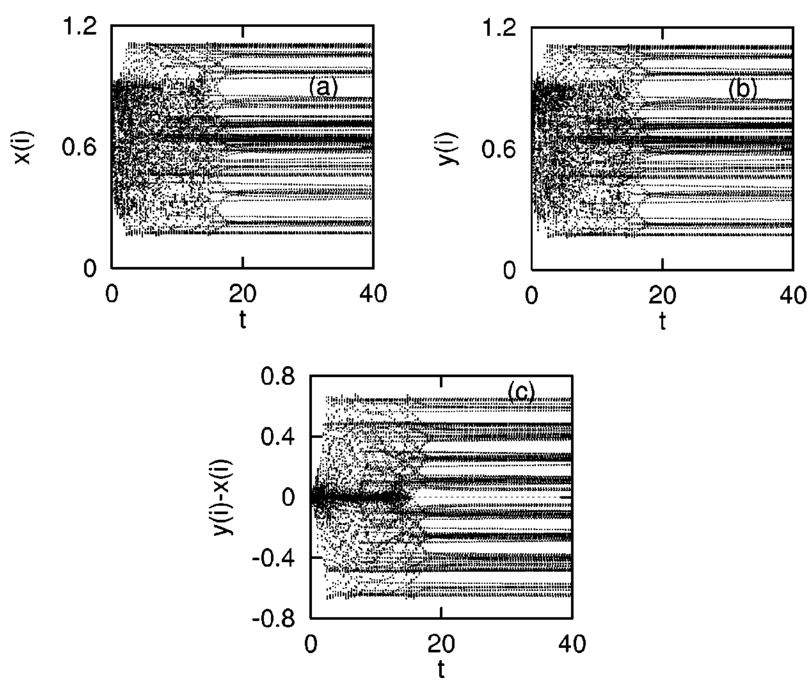

FIG. 2. Coupling behaviors when $p=0.87$. (a) $x(i)$ system; (b) $y(i)$ system; (c) error dynamics.

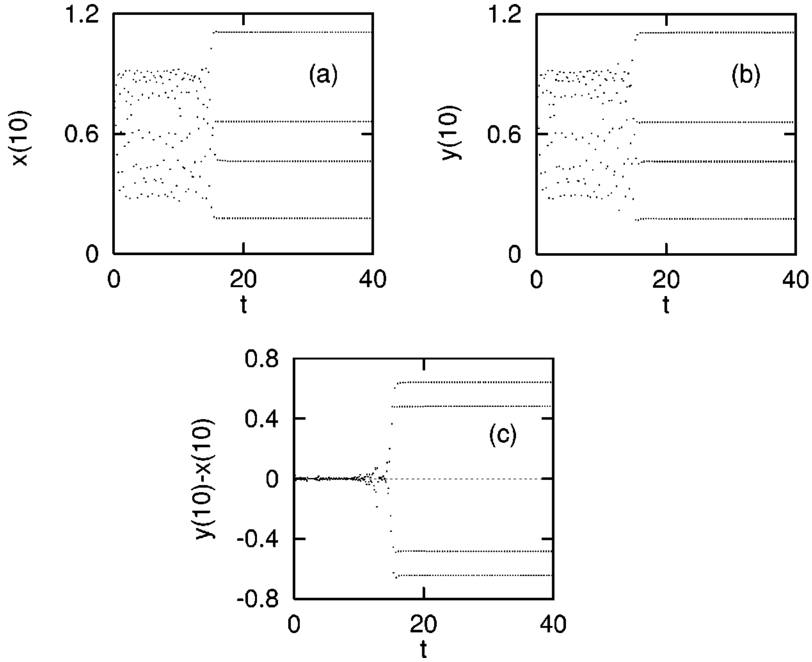

FIG. 3. Spatiotemporal evolution for one couple of sites when $p=0.87$. (a) $x(i)$ system; (b) $y(i)$ system; (c) error dynamics.

coupled systems become periodic states. If we put Fig. 3(a) and Fig. 3(b) together, we can find that the orbits of the two systems can completely overlap when $t>20$. It illustrates that there is a phase difference between the two corresponding sites when $t>15$. This conclusion is kept for other sites. So this synchronization is just GS [7–9] in which the synchronization relationship is of the form $y=\phi(\mathbf{x}(t))$. For concreteness, we take $y_n^i=x_{n-k}^{i-j}$, which is called the space-shift and time-delay synchronization. For getting more detail information we investigate the space behavior of the two coupled CML systems. Figure 4 shows the results where (a) represents the behavior of each site in $x(i)$ system when $t=30.0$, (b) the behavior of each site in $x(i)$ system when $t=30.0–60.0$, and (c) errors variation in space when $t=30.0–60.0$. From Fig. 4(a) one can see that every site stays in different states at some time. And Fig. 4(b) tells us that every site is in time-period-four and all these time-period-fours are in fixed positions. So the whole system is time-period-four.

On the other hand, for most of the coupling strengths p, the coupled systems do not become periodic states and remain in chaotic states.

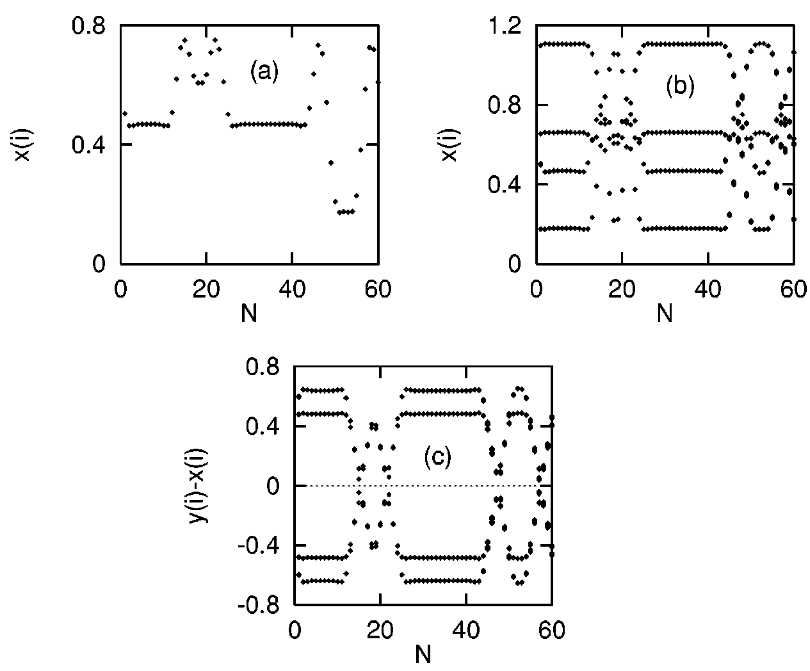

FIG. 4. Space-time diagram when $p=0.87$. (a) The behavior of each site in the $x(i)$ system when $t=30.0$; (b) the behavior of each site in the $x(i)$ system when $t=30.0–60.0$; (c) errors variation in space when $t=30.0–60.0$.

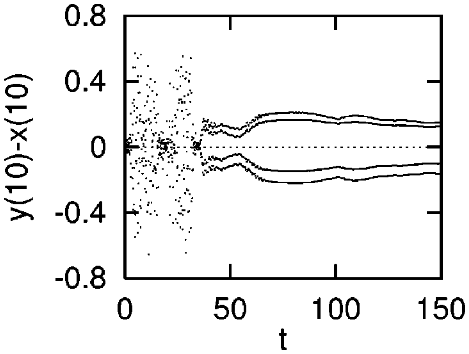

FIG. 5. Error dynamics for one couple of sites when $a_1=3.7$, $a_2=3.8$, $p=0.62$.

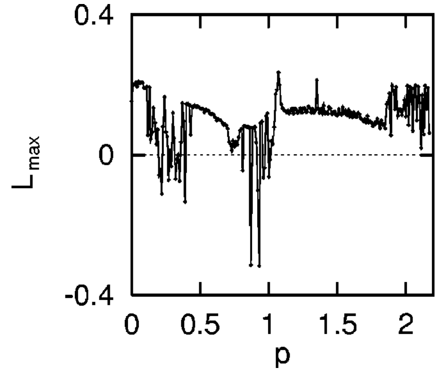

FIG. 7. Largest Lyapunov exponent spectra of the same CML models with the same parameters.

III. COUPLED SYNCHRONIZATION IN THE SAME CML MODELS WITH DIFFERENT PARAMETERS

In this section, we discuss the case of the same CML models with different parameters. We also use the function of Eq. (2) as an example and let the parameter of the $x(i)$ system be $a_1=3.7$ and the parameter of the $y(i)$ system be $a_2=3.8$, and others are the same as those in Sec. II. Considering the same coupling of Eqs. (3) and (4), our numerical simulation shows that the difference between the two CML systems do not become zero or constants but some special structures. Figure 5 shows the result of one couple of sites, where the coupling strength $p=0.62$. Obviously, the difference, $y(10)-x(10)$, becomes four curves when $t>50.0$. Comparing Fig. 5 with Fig. 3(c) one can see that they have two different points. The first is that the synchronized time of Fig. 5 is much longer than Fig. 3(c), and the second is that the difference of Fig. 3(c) is a straight line but Fig. 5 is a curve. Does the other sites have the similar behavior? Figure 6 shows the behavior of spatiotemporal synchronization, where (a) represents system $x(i)$ and (b) system $y(i)$. Comparing Fig. 6(a) with Fig. 6(b) one can see that they have some similarity. Considering Fig. 5 and Fig. 6 at the same time one can see that there is some transformation relation between the two CML systems. So they are GS.

Practically speaking, it is impossible to have two identical chaotic systems. The practical synchronization occurs between two slightly different chaotic systems. So this kind of synchronization with different parameters has special importance in practical communication.

IV. LYAPUNOV EXPONENT SPECTRA

What is the reason for the mutual coupling to induce the systems to become periodic and then IS or GS? One can give an explanation in terms of Lyapunov exponents, which measure the growth of small perturbations of the differences between the two systems. We know that under suitable coupling strength one system will run following another system if its largest conditional Lyapunov exponent is negative, and the two systems will not run in chaotic states but only some special structures if the largest Lyapunov exponent of the whole coupled systems is negative. There must be some transform relationship between the two special structures. The two systems are IS when the transform relation is identity, otherwise they are GS.

How can one calculate the largest Lyapunov exponent of the whole system? We know that the whole system is $2N$ dimensional. Now we construct an aided system nearby in the original system. The distance between the aided and original systems is $\sqrt{\Sigma_{i=1}^{2N}\delta^2 x_0(i)}$. With the evolution of time, this distance will expand along the largest eigenvalue direction. So the largest Lyapunov exponent can be obtained as follows:

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\ln\frac{\sqrt{\sum_{i=1}^{2N}\delta^2 x_t(i)}}{\sqrt{\sum_{i=1}^{2N}\delta^2 x_0(i)}}. \tag{6}$$

Figure 7 shows the results of the case of same CML models with same parameters. Figure 8 shows the results of the case of same CML models with different parameters. From Fig. 7 and Fig. 8 we know that the largest Lyapunov exponents are negative when the coupling strength p is in some narrow

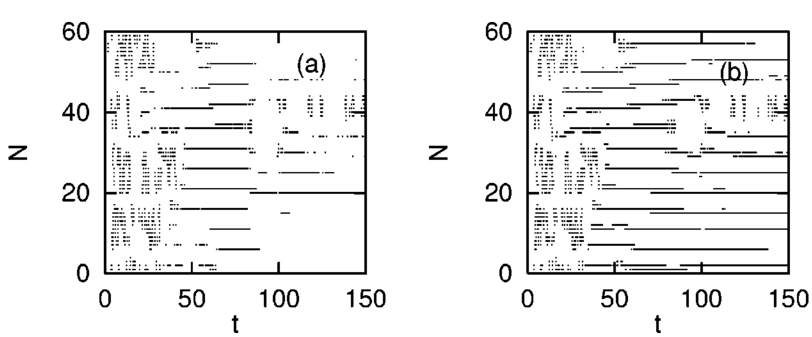

FIG. 6. Space-time diagram where $a_1=3.7$, $a_2=3.8$, $p=0.62$, and the ordinate denotes the positions of sites. Pixels are painted black if $x(i)\geqslant 0.98$ (or $y(i)\geqslant 0.98$), and left blank otherwise. (a) $x(i)$ system; (b) $y(i)$ system.

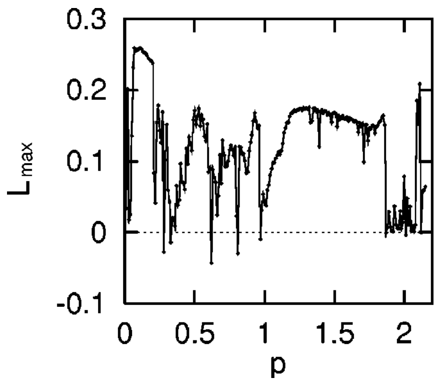

FIG. 8. Largest Lyapunov exponent spectra of the same CML models with different parameters.

intervals and positive otherwise. It means that GS can only be implemented in these narrow intervals of p. The above considered cases of $p=0.87$ in Sec. II and $p=0.62$ in Sec. III are just in these narrow intervals of p. Especially, when $p=0$, the largest Lyapunov exponents are positive in both Fig. 7 and Fig. 8. It means that the synchronization cannot be implemented for the system in isolation $(p=0)$. When coupling strength p is larger than some value (It is 2.17 for the case of same parameters, and 2.18 for the case of different parameters), the systems just have runaway solution because the coupling strength is too strong.

In conclusion, we have discussed the mutual synchronization of spatiotemporal chaos induced by nonlinear coupling. The GS will appear when the coupling strength is suitable, and the systems will have runaway solution when the coupling strength is too strong. It has some practical importance for the appearance of GS in the case of different parameters. These phenomena can be explanated by the largest Lyapunov exponent spectra.

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